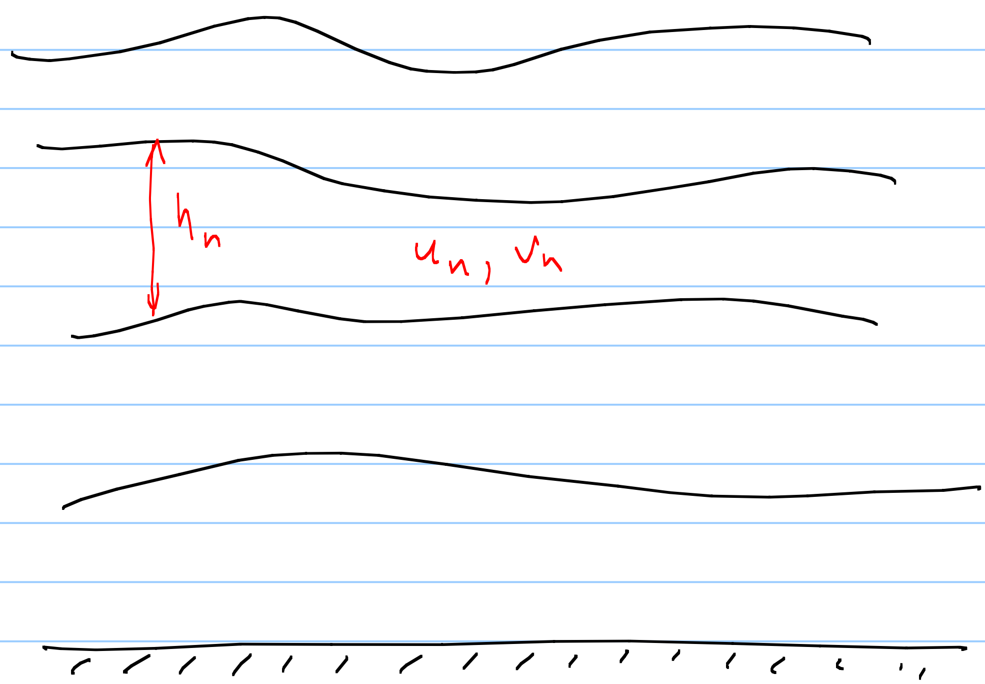
Lecture 10

Note Title 5/19/2009

The Potential Vorticity in a layered system

Consider a layered ocean, with an arbitrary number of layers:



Each layer has a thickness h_n and horizontal velocity u_n, v_n that is uniform over that layer.

The vorticity equation in each layer is:

$$\frac{D}{Dt}(f+\zeta_n) = -(f+\zeta_n)\left(\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y}\right)$$

where $$\zeta_n = \frac{\partial v_n}{\partial x} - \frac{\partial u_n}{\partial y}$$

For an inviscid fluid

the continuity equation is in each layer:

$$\frac{Dh_n}{Dt} = -h_n\left(\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y}\right)$$

These two equations can be combined:

$$\frac{1}{h_n}\frac{D(\zeta_n+f)}{Dt} = +\frac{(f+\zeta_n)}{h_n^2}\frac{Dh_n}{Dt}$$

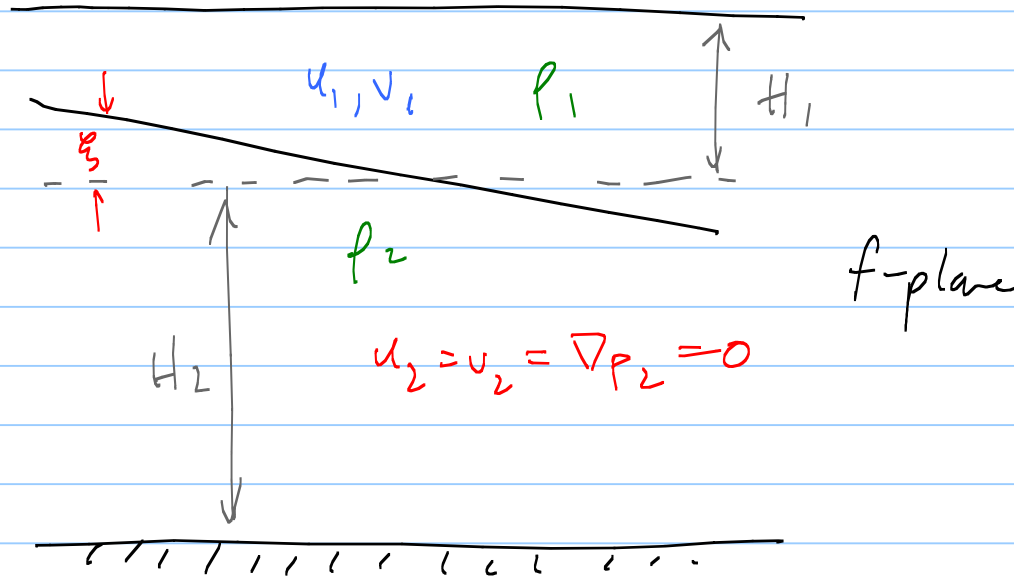
$$\frac{1}{h_n}\frac{D}{Dt}(\zeta_n+f) + (f+\zeta_n)\frac{D}{Dt}\left(\frac{1}{h_n}\right) = 0$$

$$\boxed{\Rightarrow \frac{D}{Dt}q_n = 0 \qquad q_n = \frac{f+\zeta_n}{h_n}}$$

q_n = PV in the layered system

An important consequence is that the internal structure of the density surface i.e. isopycnals shaped the distribution of PV in the ocean.

This constrains the structure of the large scale circulation and sets the properties of Rossby waves in the ocean. Showing this with a simple example:



Two layers → no flow in bottom layer:

$$\eta = -\frac{g'}{g}\xi$$

$$\frac{Du_1}{Dt} - fv_1 = g'\frac{\partial \xi}{\partial x} \qquad \frac{Dv_1}{Dt} + fu_1 = g'\frac{\partial \xi}{\partial y}$$

$$\frac{Dh_1}{Dt} = -h_1\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial x}\right)$$

Consider the case when there is a tilt in the interface:

$$\xi = \overline{\xi} + \xi'$$

$$\frac{\partial \overline{\xi}}{\partial y} = \text{const} \ \& \ \text{steady in time}$$

For the PGF associated with this tilt to be balanced there must be a corresponding zonal flow

$$u_1 = \bar{u}_1 + u_1' \qquad v_1 = v_1'$$

$$f\bar{u}_1 = g'\frac{\partial \bar{\xi}}{\partial y} \qquad \bar{u}_1 = \frac{g'}{f}\frac{\partial \bar{\xi}}{\partial y}$$

The PV in the top layer is

$$q_1 = \frac{f + \zeta_1}{H_1 - \xi} = \bar{q}_1 + q_1'$$

Assuming that $\xi', \bar{\xi} \ll H_1$, low Rossby #

$$\bar{q}_1 = \frac{f}{H_1} + \frac{\bar{\xi} f}{H_1^2} \qquad q' = \frac{\zeta_1}{H_1} + \frac{\xi' f}{H_1^2}$$

The PV equation becomes:

$$\frac{Dq_1'}{Dt} + \frac{D\bar{q}_1}{Dt} = 0 \qquad \frac{Dq_1'}{Dt} + \vec{u}_1 \cdot \nabla \bar{q}_1 = 0$$

The gradient in the mean PV is associated with the tilt in the thermocline:

$$\frac{\partial \bar{q}_1}{\partial y} = \frac{f}{H_1^2}\frac{\partial \bar{\xi}}{\partial y} = \frac{\beta_{eff}}{H_1}$$

where $\beta_{eff} = \dfrac{f}{H_1}\dfrac{\partial \bar{\xi}}{\partial y}$ an effective β associated w/ thermocline tilt

Thus the PV equation becomes:

$$\frac{Dq_1'}{Dt} + v_1 \frac{\beta_{eff}}{H_1} = 0$$

Assuming that the flow is purely geostrophic:

$$u_1' = \frac{g'}{f}\frac{\partial \xi'}{\partial y} \qquad v_1' = \frac{-g'}{f}\frac{\partial \xi'}{\partial x}$$

$$u_1' = -\partial\psi/\partial y \qquad v_1' = +\partial\psi/\partial x$$

$$\psi = -g'/f\ \xi'$$

$$q_1' = \frac{\zeta_1}{H_1} + \frac{\xi' f}{H_1^2} = \frac{1}{H_1}\left(\nabla^2\psi - \frac{1}{(L_r^{bc})^2}\psi\right)$$

$$L_r^{bc} = \frac{\sqrt{g'H_1}}{f}$$

Now $\quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{u}_1\frac{\partial}{\partial x} + J(\psi,\ )$

so the QG PV equation for this system becomes:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\hat{q}_1' + \overline{u}_1\frac{\partial \hat{q}_1'}{\partial x} + J(\psi, \hat{q}_1') + \beta_{eff}\frac{\partial\psi}{\partial x} = 0\\ &\text{where} \quad \hat{q}_1 = \nabla^2\psi - \frac{1}{(L_r^{bc})^2}\psi\end{aligned}}$$

Lets look for plane wave solutions:

$$\psi = \psi_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$+i\omega\left[l^2 + k^2 + (1/L_r^{bc})^2\right] - ik\bar{u}_1\left[l^2 + k^2 + (1/L_r^{bc})^2\right]$$

$$+ik\beta_{eff} = 0$$

$$\boxed{\omega = \bar{u}_1 k - \frac{\beta_{eff}\, k}{k^2 + l^2 + (1/L_r^{bc})^2}}$$

advection of waves by mean flow

freely propagating Rossby wave associated with mean PV gradient caused by thermocline tilt

Remember that $\bar{u}_1 = \frac{g'}{f}\frac{\partial \bar{\xi}}{\partial y}$; $\beta_{eff} = \frac{f}{H_1}\frac{\partial \bar{\xi}}{\partial y}$

$$\bar{u}_1 = \frac{g' H_1}{f^2}\beta_{eff} = (L_r^{bc})^2 \beta_{eff}$$

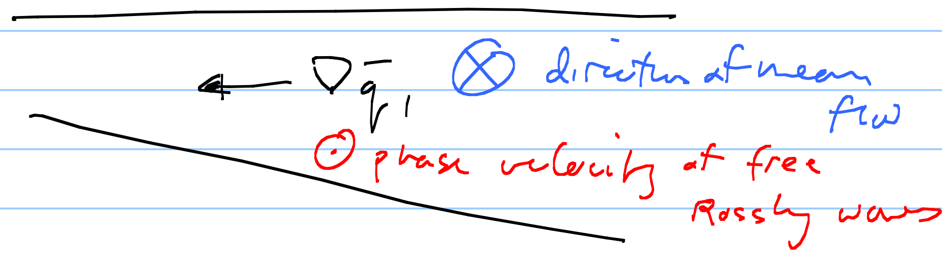
What is the phase speed of the waves?

$$c_p^x = \frac{\omega}{k} = \bar{u}_1 - \frac{\beta_{eff}}{k^2 + l^2 + (1/L_r^{bc})^2}$$

$$= \beta_{eff}(L_r^{bc})^2 - \frac{\beta_{eff}}{k^2 + l^2 + (1/L_r^2)}$$

opposite sign

Rossby waves propagate counter to mean flow



When is the phase speed of the waves equal to zero?

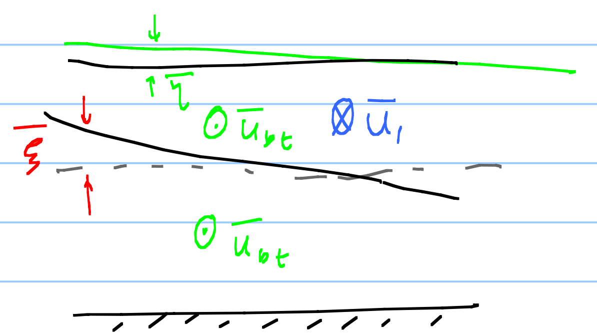
$$c_p^x = 0 = L_r^{bc\,2}(k^2+l^2)+1 - 1 = 0$$

$$\rightarrow k^2+l^2 = 0$$

waves w/ infinitely long wavelengths!

For any non-zero wavenumber the waves propagate in the direction of the mean flow. But viewed moving with the mean flow the waves propagate to the east.

What could we do to make the waves stationary? What if we added a slight tilt to the free surface that generates a flow counter to $\bar{u}_1$.



How big does $\frac{\partial \bar{\eta}}{\partial y}$ have to be for

$$|\overline{u_{bt}}| \simeq |\bar{u}_1| \qquad |\overline{u_{bt}}| = \left| -\frac{g}{f}\frac{\partial \bar{\eta}}{\partial y} \right| = |\bar{u}_1| = \left| \frac{g'}{f}\frac{\partial \bar{\xi}}{\partial y} \right|$$

$$\frac{\partial \bar{\eta}}{\partial y} = \frac{g'}{g}\frac{\partial \bar{\xi}}{\partial y} \rightarrow \bar{\eta} \sim \frac{g'}{g}\bar{\xi}$$

So $\bar{\eta} << \bar{\xi}$ so the extra tilt in the free surface won't really affect the PV gradient but it will generate a mean flow. Any tilt in the free surface that exceeds

$$\left|\frac{\partial \bar{\eta}}{\partial y}\right| > \frac{g'}{g}\frac{\partial \bar{\xi}}{\partial y}$$

will reduce the strength of the mean flow from

$$|\bar{u}_1| = \beta_{eff}\left(L_r^{bc}\right)^2$$

i.e $\quad |\bar{u}_1| < \beta_{eff}\left(L_r^{bc}\right)^2$

So that the phase speed

$$c_p^x = \bar{u}_1 - \frac{\beta_{eff}}{k^2 + l^2 + (1/L_r^{bc})^2}$$

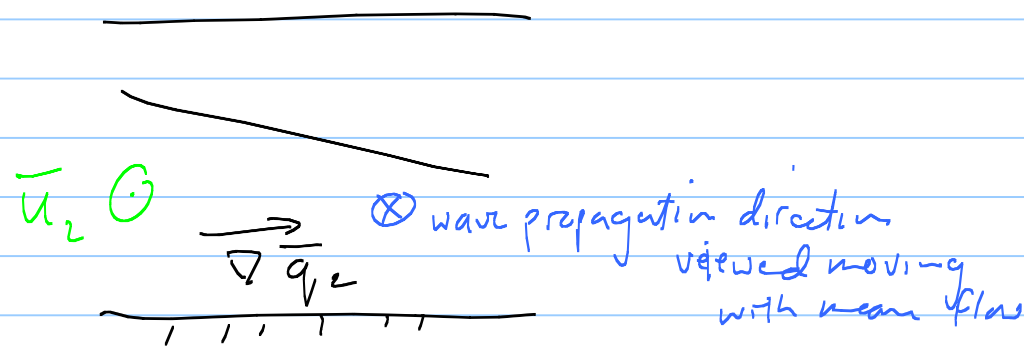
will go to zero for non-zero (k^2+l^2)

$\rightarrow$ there are some waves that are stationary

This has important implication for the development of instabilities of the system known as baroclinic instability.

To see this, if we were to relax the condition that $u_2' = v_2' = 0$ you can convince yourself that in the lower layer there will be Rossby waves that when viewed moving with the mean flow $\bar{u}_2$ will propagate to the west since the mean PV gradient in the lower layer

$$\frac{\partial \bar{q}_2}{\partial y} > 0$$



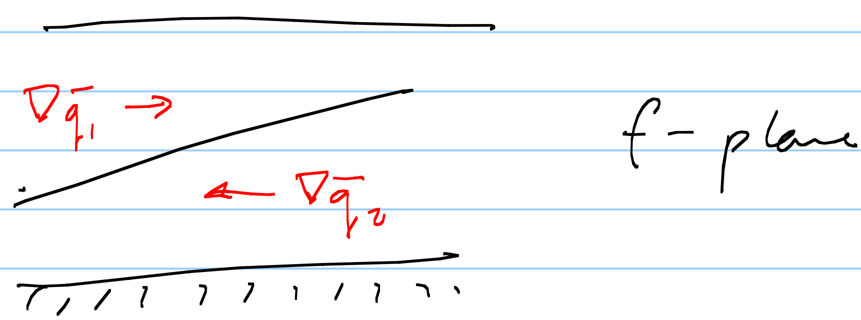
But by adding the extra tilt to the free surface this create a mean flow that is counter to the direction of propagation of the mean flow → can allow for stationary waves as well in lower layer.

When we consider waves in both layers, this slowing down of the waves by the mean flow can give rise to a phase locking of the waves in the upper and lower layer. The waves can thus couple and generate flows in both layers that instead of restoring fluid parcel displacements to their original location (as was the case for the barotropic Rossby waves) the coupled motions can push fluid parcel away from their original locations and thus lead to an instability. This instability is known as baroclinic instability.

The property of the mean flow that was necessary for this instability to occur was that the PV gradient in the two layers was of opposite sign i.e.

$$\mathrm{sgn}\left(\frac{d\bar{q}_1}{dy}\right) = -\,\mathrm{sgn}\left(\frac{d\bar{q}_2}{dy}\right)$$

so that the waves can phase lock in the presence of the baroclinic mean flow.

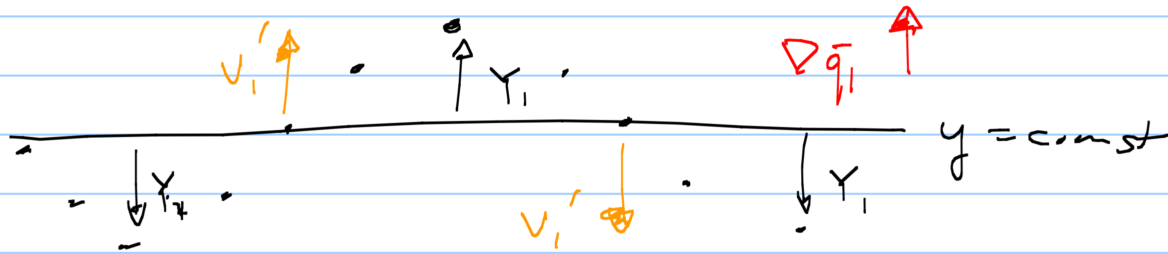


Any two layer system with a tilt in the interface on an f-plane will satisfy this necessary condition for instability.

What is the mechanism for instability?

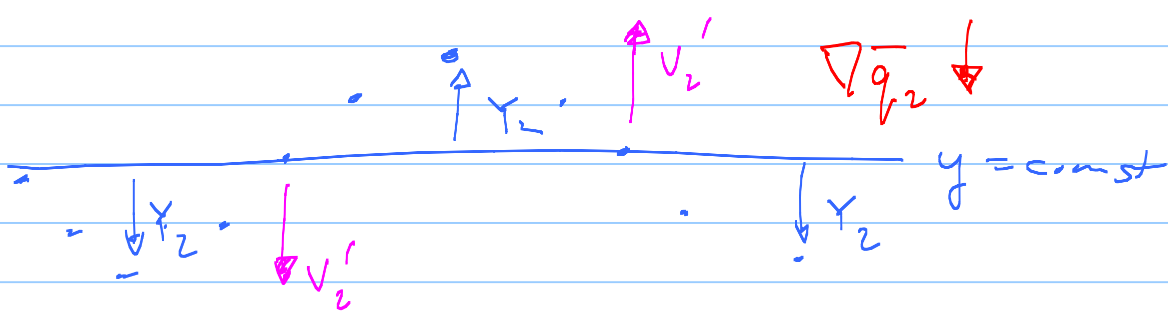
For the system where $\frac{\partial \bar{q}_1}{\partial y} > 0 \quad \frac{\partial \bar{q}_2}{\partial y} < 0$

In the upper layer:



Fluid parcel displacements Y_1 will yield PV anomalies q_1' that will drive velocity anomalies v_1' that are 90° out of phase from Y_1. This will cause the parcel displacements to shift to the left, i.e. propagate, they won't cause Y_1 to grow in time.

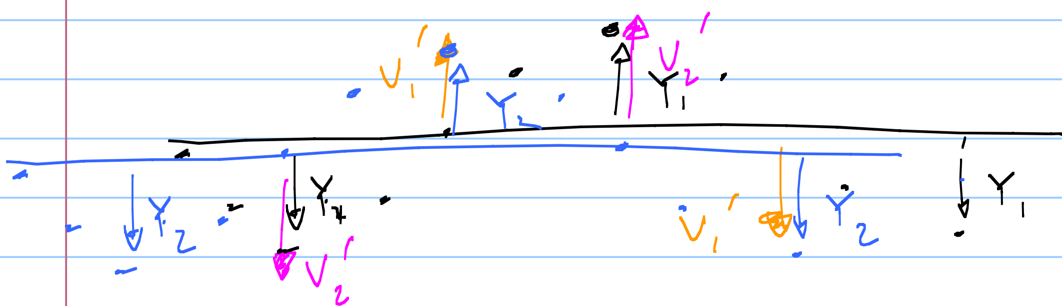
In the lower layer



Since the PV gradient is the opposite sign, the parcel displacement Y_2 will drive PV anomalies and velocity anomalies of the opposite sense as in the top layer.

But v_2' is still $90°$ out of phase with ψ_2 and so the velocity field will only lead to propagation not growth of ψ_2.

Now if we let the flow fields in the upper and lower layers interact and if ψ_2 was shifted $90°$ to the left of ψ_1:



The flow in the lower layer would be in phase with ψ_1 and thus cause ψ_1 to grow in time.

Similarly the flow from the upper layer v_1 would be in phase with ψ_2 thus causing it to grow in time.

These are the ingredients necessary for instability. But the waves in both layers must be phase locked. Hence the opposite sign PV gradients is key to the instability.